

# SYMMETRIC 1-DEPENDENT COLORINGS OF THE INTEGERS

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**ABSTRACT.** In a recent paper by the same authors, we constructed a stationary 1-dependent 4-coloring of the integers that is invariant under permutations of the colors. This was the first stationary  $k$ -dependent  $q$ -coloring for any  $k$  and  $q$ . When the analogous construction is carried out for  $q > 4$  colors, the resulting process is not  $k$ -dependent for any  $k$ . We construct here a process that is symmetric in the colors and 1-dependent for every  $q \geq 4$ . The construction uses a recursion involving Chebyshev polynomials evaluated at  $\sqrt{q}/2$ .

## 1. INTRODUCTION

By a (proper)  $q$ -coloring of the integers, we mean a sequence  $(X_i : i \in \mathbb{Z})$  of  $[q]$ -valued random variables satisfying  $X_i \neq X_{i+1}$  for all  $i$  (where  $[q] := \{1, \dots, q\}$ ). The coloring is said to be stationary if the (joint) distribution of  $(X_i : i \in \mathbb{Z})$  agrees with that of  $(X_{i+1} : i \in \mathbb{Z})$ , and  $k$ -dependent if the families  $(X_i : i \leq m)$  and  $(X_i : i > m + k)$  are independent of each other for each  $m$ . In [2], we gave a construction of a stationary 1-dependent 4-coloring of the integers that is invariant under permutations of the colors. When the same construction is carried out for  $q > 4$  colors, the resulting distribution is not  $k$ -dependent for any  $k$ . Of course, the 1-dependent 4-coloring is also a 1-dependent  $q$ -coloring for every  $q > 4$ , and one may obtain other 1-dependent  $q$ -colorings by splitting a color into further colors using an independent source of randomness. However, these colorings are not symmetric in the colors. We give here a modification of the process of [2] that is symmetric in the colors and 1-dependent for every  $q \geq 4$ . Here is our main result.

**Theorem 1.** *For each integer  $q \geq 4$ , there exists a stationary 1-dependent  $q$ -coloring of the integers that is invariant in law under permutations of the colors and under the reflection  $(X_i : i \in \mathbb{Z}) \mapsto (X_{-i} : i \in \mathbb{Z})$ .*

Our construction is given in the next section. Sections 3 and 4 provide some preliminary results and the proof of Theorem 1 respectively.

## 2. THE CONSTRUCTION

For  $x = (x_1, x_2, \dots, x_n) \in [q]^n$ , we will write  $P(x) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n)$ . To motivate the construction, we begin by noting that the finite-dimensional distributions  $P$  of the 4-coloring in [2] are defined recursively by  $P(\emptyset) = 1$  and

$$(1) \quad P(x) = \frac{1}{2(n+1)} \sum_{i=1}^n P(\hat{x}_i)$$

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for proper  $x \in [4]^n$ , where  $\hat{x}_i$  is obtained from  $x$  by deleting the  $i$ th entry in  $x$ . Of course, even if  $x$  is proper,  $\hat{x}_i$  may not be. So the definition is completed by setting  $P(x) = 0$  for  $x$ 's that are not proper.

For general  $q \geq 4$ , we will now allow the coefficients in the defining sum to depend on  $i$  as well as  $n$ . Considering many special cases, and the constraints imposed by the 1-dependence requirement, we were led to define

$$(2) \quad P(x) = \frac{1}{D(n+1)} \sum_{i=1}^n C(n-2i+1)P(\hat{x}_i)$$

for proper  $x \in [4]^n$ , in terms of two sequences  $C$  and  $D$ . Again motivated by computations in special cases, we take

$$\begin{aligned} C(n) &= T_n(\sqrt{q}/2), & n \geq 0; \\ D(n) &= \sqrt{q} U_{n-1}(\sqrt{q}/2), & n \geq 1, \end{aligned}$$

where  $T_n$  and  $U_n$  are the Chebyshev polynomials of the first and second kind respectively.

There are several standard equivalent definitions of Chebyshev polynomials. One is

$$(3) \quad T_n(u) = \cosh(nt) \quad \text{and} \quad U_n(u) = \frac{\sinh[(n+1)t]}{\sinh(t)}, \quad \text{where } u = \cosh(t).$$

A variant definition using trigonometric functions (e.g. (22:3:3-4) of [3]) is easily seen to be equivalent by taking  $t$  imaginary; the hyperbolic function version is convenient for arguments  $u \geq 1$ . Another definition is

$$T_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} u^{n-2k} (u^2 - 1)^k \quad \text{and} \quad U_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} u^{n-2k} (u^2 - 1)^k.$$

That this is equivalent to (3) follows from e.g. (22:3:1-2) of [3].

If  $x$  is not a proper coloring, we take  $P(x) = 0$  as before. We extend both sequences  $C$  and  $D$  to all integer arguments by taking  $C(n)$  and  $D(n)$  to be even and odd functions of  $n$  respectively (in accordance with (3)).

Observe that  $C(n)$  and  $D(n)$  are strictly positive for  $q \geq 4$  and  $n \geq 1$ , and therefore  $P(x)$  is strictly positive for all proper  $x$ . Note also that  $C(n-2i+1)/D(n+1)$  is rational; therefore so is  $P(x)$ . (The factors of  $\sqrt{q}$  cancel). When  $q = 4$  we have  $C(n) = 1$  and  $D(n) = 2n$ , and so (2) reduces to (1) in this case. As we will see, the fact that the coefficients in (2) depend on  $i$  substantially complicates the verifications of the required properties of  $P$ .

Here are a few examples of cylinder probabilities generated by (2).

$$\begin{aligned} P(1) &= \frac{1}{q}, & P(12) &= \frac{1}{q(q-1)}, & P(121) &= \frac{1}{q^2(q-1)}, & P(123) &= \frac{1}{q^2(q-2)}, \\ P(1212) &= \frac{q-3}{q^2(q-1)(q^2-3q+1)}, & P(1234) &= \frac{1}{q^2(q^2-3q+1)}. \end{aligned}$$

## 3. PRELIMINARY RESULTS

Chebyshev polynomials satisfy a number of standard identities. They lead to identities satisfied by the sequences  $C$  and  $D$ . The first three in the proposition below are examples of this. The fourth is a consequence of the third one. Before stating them, we record some values of  $C$  and  $D$  to facilitate checking computations here and later.

$$C(0) = 1, \quad C(1) = \frac{\sqrt{q}}{2}, \quad C(2) = \frac{q-2}{2}, \quad C(3) = \frac{\sqrt{q}(q-3)}{2}, \quad C(4) = \frac{q^2-4q+2}{2}.$$

$$D(0) = 0, \quad D(1) = \sqrt{q}, \quad D(2) = q, \quad D(3) = \sqrt{q}(q-1), \quad D(4) = q(q-2).$$

**Proposition 2.** *For  $j, k, \ell, m, n \in \mathbb{Z}$ , the following identities hold.*

$$(4) \quad 2C(m)C(n) = C(m+n) + C(n-m).$$

$$(5) \quad \frac{q-4}{2q}D(m)D(n) = C(m+n) - C(n-m).$$

$$(6) \quad 2C(m)D(n) = D(m+n) + D(n-m).$$

$$(7) \quad C(j+k)D(k+\ell) = C(k)D(j+k+\ell) - C(\ell)D(j).$$

*Proof.* The first three parts are immediate consequences of (22:5:5-7) in [3], or 22.7.24-26 in [1], if  $m$  and  $n$  are nonnegative. None of the identities is changed by changing the sign of either  $m$  or  $n$ . Therefore, they hold for all  $m$  and  $n$ . Alternatively, the identities may be checked directly from (3) using the product formulae for hyperbolic functions. For (7), replace the products of  $C$ 's and  $D$ 's by sums of  $D$ 's using (6), and then use the fact that  $D$  is an odd function.  $\square$

Next we verify some identities that involve both the sequences  $C$  and  $D$  and the measure  $P$  defined by (2). For the statement of the second part of the next result, let

$$Q(x) = \frac{1}{D(n+1)} \sum_{i=1}^n C(2i)P(\hat{x}_i) \quad \text{and} \quad Q^*(x) = \frac{1}{D(n+1)} \sum_{i=1}^n C(2n-2i+2)P(\hat{x}_i)$$

for  $x \in [q]^n$ . The first part of Proposition 3 is needed in proving the second part, which plays a key role in the proof of consistency and 1-dependence of  $P$ . Note the similarity between the left side of (8) and the right side of (2).

**Proposition 3.** *If  $n \geq 1$ , and  $x$  is a proper coloring of length  $n$ , then*

$$(8) \quad \sum_{i=1}^n D(n-2i+1)P(\hat{x}_i) = 0;$$

$$(9) \quad Q(x) = Q^*(x) = P(x)C(n+1).$$

*Proof.* For the first statement, let  $R$  be the set of proper colorings, and  $\hat{x}_A$  be obtained by deleting the entries  $x_i$  for  $i \in A$  from  $x$ . The proof of (8) is by induction on  $n$ , the length of  $x$ . The identity is easily seen to be true if  $n \leq 2$ . Suppose that (8) is true for all  $x$  of length  $n-1$ , and let  $x \in R$  have length  $n$ . For those  $i$  with  $\hat{x}_i \in R$ , applying (8) gives

$$(10) \quad \sum_{j=1}^{i-1} D(n-2j)P(\hat{x}_{i,j}) + \sum_{j=i+1}^n D(n-2j+2)P(\hat{x}_{i,j}) = 0.$$

On the other hand, if  $\hat{x}_i \notin R$ , then  $1 < i < n$  and

$$(11) \quad P(\hat{x}_{i,j}) = 0 \text{ if } |j - i| > 1 \text{ and } P(\hat{x}_{i-1,i}) = P(\hat{x}_{i,i+1}).$$

The left side of (8) for  $x$  can be written, using the definition of  $P(\hat{x}_i)$  and then (6), as

$$(12) \quad \begin{aligned} &= \frac{1}{D(n)} \sum_{\substack{1 \leq i \leq n: \\ \hat{x}_i \in R}} D(n - 2i + 1) \left[ \sum_{1 \leq j < i} C(n - 2j) P(\hat{x}_{i,j}) + \sum_{i < j \leq n} C(n - 2j + 2) P(\hat{x}_{i,j}) \right] \\ &= \frac{1}{2D(n)} \sum_{\substack{1 \leq j < i \leq n: \\ \hat{x}_i \in R}} [D(2n - 2i - 2j + 1) + D(2j - 2i + 1)] P(\hat{x}_{i,j}) \\ &\quad + \frac{1}{2D(n)} \sum_{\substack{1 \leq i < j \leq n: \\ \hat{x}_i \in R}} [D(2n - 2i - 2j + 3) + D(2j - 2i - 1)] P(\hat{x}_{i,j}). \end{aligned}$$

Rearranging, and ignoring the  $2D(n)$  in the denominator, gives

$$(13) \quad \begin{aligned} &\sum_{i=1}^n \mathbf{1}[\hat{x}_i \in R] \left[ \sum_{j=1}^{i-1} [D(2n - 2i - 2j + 1) + D(2j - 2i + 1)] P(\hat{x}_{i,j}) \right. \\ &\quad \left. + \sum_{j=i+1}^n [D(2n - 2i - 2j + 3) + D(2j - 2i - 1)] P(\hat{x}_{i,j}) \right]. \end{aligned}$$

We must show that (10) and (11) imply that (13) is zero.

We would like to write (13) as a linear combination of expressions that vanish because of (10) and (11) as follows.

$$(14) \quad \sum_{\substack{1 \leq i \leq n: \\ \hat{x}_i \in R}} \alpha_i \left[ \sum_{j=1}^{i-1} D(n - 2j) P(\hat{x}_{i,j}) + \sum_{j=i+1}^n D(n - 2j + 2) P(\hat{x}_{i,j}) \right] + \sum_{\substack{1 \leq i \leq n: \\ \hat{x}_i \in R}} \sum_{j=1}^n \beta_{i,j} P(\hat{x}_{i,j}),$$

where  $\beta_{i,i} = \beta_{i,i-1} + \beta_{i,i+1} = 0$ . If  $1 \leq i < j \leq n$ , the coefficient of  $P(\hat{x}_{i,j})$  in (13) is

$$(15) \quad \begin{aligned} &\mathbf{1}[\hat{x}_j \in R] [D(2n - 2i - 2j + 1) + D(2i - 2j + 1)] \\ &+ \mathbf{1}[\hat{x}_i \in R] [D(2n - 2i - 2j + 3) + D(2j - 2i - 1)]. \end{aligned}$$

The coefficient of  $P(\hat{x}_{i,j})$  in (14) is

$$(16) \quad \mathbf{1}[\hat{x}_j \in R] \alpha_j D(n - 2i) + \mathbf{1}[\hat{x}_i \in R] \alpha_i D(n - 2j + 2) + \mathbf{1}[\hat{x}_i \notin R] \beta_{i,j} + \mathbf{1}[\hat{x}_j \notin R] \beta_{j,i}.$$

We need to choose the  $\alpha$ 's and  $\beta$ 's so that (15) and (16) agree. If  $\hat{x}_i, \hat{x}_j \in R$ , this says

$$D(2n - 2i - 2j + 1) + D(2n - 2i - 2j + 3) = \alpha_j D(n - 2i) + \alpha_i D(n - 2j + 2)$$

since  $D$  is an odd function. It may sound unreasonable to expect to solve this system, since there are  $n$  unknowns and  $\binom{n}{2}$  equations. However,  $D$  satisfies relations that make this possible. Solving the equations for small  $n$  suggests trying  $\alpha_i = 2C(n - 2i + 1)$ . The fact that this choice solves these equations for all choices of  $n, i, j$  then follows from (6)

and the fact that  $D$  is odd. If  $\hat{x}_i \notin R$  and  $\hat{x}_j \notin R$ , (15) and (16) agree if  $\beta_{i,j} + \beta_{j,i} = 0$ . If  $\hat{x}_i \in R$  and  $\hat{x}_j \notin R$ , they agree if

$$D(2n - 2i - 2j + 3) + D(2j - 2i - 1) = \alpha_i D(n - 2j + 2) + \beta_{j,i}.$$

Using (6) again gives  $\beta_{j,i} = 2D(2j - 2i - 1)$ . Similarly, if  $\hat{x}_i \notin R$  and  $\hat{x}_j \in R$ , they agree if  $\beta_{i,j} = 2D(2i - 2j + 1)$ . With these choices,  $\beta$  is anti-symmetric, and  $\beta_{k,k-1} = 2D(1)$  and  $\beta_{k,k+1} = 2D(-1)$ , so  $\beta_{k,k-1} + \beta_{k,k+1} = 0$  as required. This completes the induction argument.

For (9), consider the case of  $Q$  first. Use the definition of  $P$  to write the right side of (9) as

$$\frac{C(n+1)}{D(n+1)} \sum_{i=1}^n C(n-2i+1)P(\hat{x}_i).$$

Using (4), this becomes

$$\frac{1}{2D(n+1)} \sum_{i=1}^n C(2n-2i+2)P(\hat{x}_i) + \frac{1}{2}Q(x).$$

Therefore, we need to prove that

$$\sum_{i=1}^n [C(2n-2i+2) - C(2i)] P(\hat{x}_i) = 0.$$

But by (5), this follows from (8). The proof for  $Q^*$  is similar.  $\square$

#### 4. PROOF OF THE MAIN RESULT

We will often write  $x_1 x_2 \cdots x_n$  instead of  $(x_1, x_2, \dots, x_n)$  below. If  $x \in [q]^m$  and  $y \in [q]^n$ , let  $xy$  denote the word  $x_1 \cdots x_m y_1 \cdots y_n \in [q]^{m+n}$ .

*Proof of Theorem 1.* We first need to show that the finite dimensional distributions defined in (2) are consistent, i.e., that

$$(17) \quad \sum_{a \in [q]} P(xa) = P(x), \quad x \in [q]^n, \quad n \geq 0.$$

This is true if  $x$  is not proper, since then  $xa$  is also not proper, and so both sides vanish. For proper  $x$ , the proof is by induction on  $n$ . Note that for  $a \in [q]$ ,

$$P(a) = \frac{C(0)}{D(2)} = \frac{1}{q},$$

so  $\sum_{a \in [q]} P(a) = 1$ . This gives (17) for  $n = 0$ . Suppose it holds for all  $x \in [q]^{n-1}$  with  $n \geq 1$ . Then for proper  $x \in [q]^n$ , using the induction hypothesis in the second equality,

$$\begin{aligned} \sum_{a \in [q]} P(xa) &= \sum_{a \neq x_n} \frac{1}{D(n+2)} \left[ \sum_{i=1}^n C(n-2i+2)P(\hat{x}_i a) + C(-n)P(x) \right] \\ &= \frac{1}{D(n+2)} \left[ \sum_{i=1}^n C(n-2i+2)P(\hat{x}_i) - C(-n+2)P(x) + (q-1)C(-n)P(x) \right]. \end{aligned}$$

The middle term in the second line accounts for the missing term  $a = x_n$  when the inductive hypothesis is applied to the case  $i = n$  (since  $\widehat{x}_n x_n = x$ ). Using  $(j, k, \ell) = (1, n - 2i + 1, 2i)$  in (7) gives

$$\frac{C(n - 2i + 2)}{D(n + 2)} = \frac{C(n - 2i + 1)}{D(n + 1)} - \frac{C(2i)D(1)}{D(n + 2)D(n + 1)}.$$

Therefore

$$\sum_{a \in [q]} P(xa) = P(x) - \frac{Q(x)}{D(n + 2)} - \frac{C(n - 2)}{D(n + 2)} P(x) + (q - 1) \frac{C(n)}{D(n + 2)} P(x).$$

This is  $P(x)$ , as required, by (9) and the fact that

$$(q - 1)C(n) = C(n - 2) + C(n + 1)D(1),$$

which is obtained by taking  $(j, k, \ell) = (2, -n, n + 1)$  in (7), and then canceling a factor of  $\sqrt{q}$ .

Invariance of the measure under permutations of colors and translations is immediate from the definition. Invariance under reflection amounts to checking  $P(x) = P(x_n \cdots x_1)$ , which follows from the fact that the coefficients of  $\widehat{x}_i$  and  $\widehat{x}_{n-i+1}$  in (2), which are  $C(n - 2i + 1)$  and  $C(-n + 2i - 1)$  respectively, are equal by the symmetry of  $C$ .

For 1-dependence, we need to show that for  $x \in [q]^m$  and  $y \in [q]^n$  with  $m, n \geq 0$ ,

$$P(x * y) = P(x)P(y),$$

where the  $*$  means that there is no constraint at the single site between  $x$  and  $y$ . This is again true if  $x$  or  $y$  is not proper since then both sides are zero. For proper  $x$  and  $y$ , the proof is by induction, but now on  $m + n$ . The statement is immediate if  $m = 0$  or  $n = 0$ . So, we take  $m \geq 1$  and  $n \geq 1$ .

There are two cases, according to whether or not  $xy$  is a proper coloring, i.e., whether  $x_m$  and  $y_1$  are equal or different. Assume first that  $x_m = y_1$ . Without loss of generality, take their common value to be 1. Then using the definition of  $P$ , including the fact that  $P(xy) = 0$ ,

$$\begin{aligned} (18) \quad P(x * y) &= \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \sum_{a \neq 1} \left[ \sum_{i=1}^m C(n + m - 2i + 2) P(\widehat{x}_i a y) \right. \\ &\quad \left. + C(n - m) P(xy) + \sum_{j=1}^n C(n - m - 2j) P(x a \widehat{y}_j) \right] \\ &= \frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^m C(n + m - 2i + 2) P(\widehat{x}_i * y) + \sum_{j=1}^n C(n - m - 2j) P(x * \widehat{y}_j) \right]. \end{aligned}$$

Using the induction hypothesis, this becomes

$$P(x * y) = \frac{1}{D(n + m + 2)} \left[ P(y) \sum_{i=1}^m C(n + m - 2i + 2) P(\widehat{x}_i) + P(x) \sum_{j=1}^n C(n - m - 2j) P(\widehat{y}_j) \right].$$

Taking  $(j, k, l) = (n, m - 2i + 1, i)$  in (7) gives

$$\frac{C(n + m - 2i + 2)}{D(n + m + 2)} = \frac{C(m - 2i + 1)}{D(m + 1)} - \frac{C(2i)D(n + 1)}{D(m + 1)D(n + m + 2)}.$$

Similarly,

$$\frac{C(m + 2j - n)}{D(n + m + 2)} = \frac{C(2j - n - 1)}{D(n + 1)} - \frac{C(2n - 2j + 2)D(m + 1)}{D(n + 1)D(n + m + 2)}.$$

Therefore, since  $C(\cdot)$  is even,

$$P(x * y) = P(y) \left[ P(x) - \frac{D(n + 1)}{D(n + m + 2)} Q(x) \right] + P(x) \left[ P(y) - \frac{D(m + 1)}{D(n + m + 2)} Q^*(y) \right].$$

By (9),

$$P(x * y) = P(x)P(y) \left[ 2 - \frac{C(m + 1)D(n + 1) + C(n + 1)D(m + 1)}{D(n + m + 2)} \right].$$

Taking  $(j, k, l) = (n, m - 2i + 1, i)$  in (7), we see that the expression in brackets above is 1, as required.

Assume now that  $x_m \neq y_1$ , say  $x_m = 1$  and  $y_1 = 2$ . Then

$$\begin{aligned} (19) \quad P(x * y) &= \sum_{a \in [q]} P(xay) = \frac{1}{D(n + m + 2)} \sum_{a \neq 1, 2} \left[ \sum_{i=1}^m C(n + m - 2i + 2) P(\widehat{x}_i ay) \right. \\ &\quad \left. + C(n - m) P(xy) + \sum_{j=1}^n C(n - m - 2j) P(xa \widehat{y}_j) \right] \\ &\quad \frac{1}{D(n + m + 2)} \left[ \sum_{i=1}^m C(n + m - 2i + 2) P(\widehat{x}_i * y) + \sum_{j=1}^n C(n - m - 2j) P(x * \widehat{y}_j) \right] \end{aligned}$$

as in the previous case. However, in the previous case, the term  $P(xy)$  dropped out because  $xy$  was not a proper coloring. In this case, the term  $(q - 2)C(n - m)P(xy)$  is cancelled by the terms  $-P(xy)C(n - m + 2)$  and  $-P(xy)C(n - m - 2)$ , which arise from

$$\sum_{a \neq 1, 2} P(\widehat{x}_m ay) = P(\widehat{x}_m * y) - P(xy) \text{ and } \sum_{a \neq 1, 2} P(xa \widehat{y}_1) = P(x * \widehat{y}_1) - P(xy).$$

The fact that the overall coefficient of  $P(xy)$  vanishes is a consequence (4) with  $m = 2$ , since  $2C(2) = q - 2$ . The rest of the proof is the same as in the case  $x_m = y_1$  above.  $\square$

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